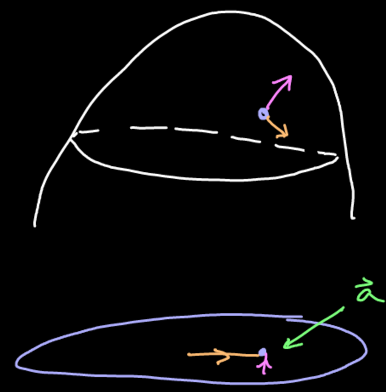
Directional Derivative

Idea: The derivative of a single variable function represents the instantaneous rate of change over time.

When the domain is "bigger" (i.e. has higher dimension), the instantaneous change can depend on the direction in which we approach the point (as with limits).



Thus, a priori we want to measure changes in a given direction.

We'll do this exactly the same way as in Calculus 1...

We see how small changes in the direction of $\vec{v}$ affect the output of f. In mathese...

Def^n: Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $\vec{p} \in D$, and $\vec{v} \in \mathbb{R}^n$ a unit vector. The directional derivative of f at $\vec{p}$ in the direction of $\vec{v}$ is

$$D_{\vec{v}} f(\vec{p}) = \lim_{h \to 0^+} \frac{f(\vec{p} + h\vec{v}) - f(\vec{p})}{h}.$$

NB: For $\vec{v} \neq 0$, one can use the corresponding unit vector $\frac{\vec{v}}{|\vec{v}|}$. In the limit, the result is a scaling factor of $\frac{1}{|\vec{v}|}$ ...

Ex: Compute $D_{\vec{u}} f(\vec{p})$ for ...

① $f(x,y) = xy$, $\vec{p} = \langle 2, 4 \rangle$, $\vec{u} = \langle 2, -1 \rangle$

Sol: $$D_{\frac{\vec{u}}{|\vec{u}|}} f(\vec{p}) = \lim_{h\to 0^+} \frac{f(\vec{p}+h\vec{v}) - f(\vec{p})}{h} \cdot \frac{1}{|\vec{u}|}$$

$$= \lim_{h\to 0^+} \frac{f(2+2h, 4-h) - f(2,4)}{h} \cdot \frac{1}{\sqrt{2^2+1^2}}$$

$$= \lim_{h\to 0^+} \frac{(2+2h)(4-h) - 2\cdot 4}{h} \cdot \frac{1}{\sqrt{5}}$$

$$= \lim_{h\to 0^+} \frac{8 - 2h + 8h - 2h^2 - 8}{h} \cdot \frac{1}{\sqrt{5}}$$

$$= \lim_{h\to 0^+} \frac{h(6-2h)}{h\sqrt{5}}$$

$$= \lim_{h\to 0^+} \frac{(6-2h)}{\sqrt{5}} = \frac{6-2\cdot 0}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$ //

② $f(x,y) = x\sqrt{y}$, $\vec{p} = \langle -1, 1 \rangle$, $\vec{u} = \langle 1, 1 \rangle$

Sol: $$D_{\frac{\vec{u}}{|\vec{u}|}} f(\vec{p}) = \lim_{h\to 0^+} \frac{f(\vec{p}+h\vec{u}) - f(\vec{p})}{h} \cdot \frac{1}{|\vec{u}|}$$

$$= \lim_{h\to 0^+} \frac{f(-1+h, 1+h) - f(-1,1)}{h} \cdot \frac{1}{\sqrt{1^2+1^2}}$$

$$= \lim_{h\to 0^+} \frac{(-1+h)\sqrt{1+h} - (-1)\sqrt{1}}{h} \cdot \frac{1}{\sqrt{2}}$$

$$= \lim_{h\to 0^+} \frac{(-1+h)\sqrt{1+h} + 1}{\sqrt{2}\, h} \cdot \frac{1 - (h-1)\sqrt{1+h}}{1 - (h-1)\sqrt{1+h}}$$

$$= \lim_{h\to 0^+} \frac{1 - (h-1)^2(h+1)}{\sqrt{2}\, h\,(1 - (h-1)\sqrt{1+h})}$$

$$(h^2 - 2h + 1)\cdot(h+1) = h^3 - h^2 - h + 1$$

$$= \lim_{h\to 0^+} \frac{1 - (h^3 - h^2 - h + 1)}{\sqrt{2}\, h\,(1 - (h-1)\sqrt{1+h})}$$

$$= \lim_{h\to 0^+} \frac{\cancel{h}(-h^2 + h + 1)}{\sqrt{2}\,\cancel{h}\,(1 - (h-1)\sqrt{1+h})} = \frac{-0^2 + 0 + 1}{\sqrt{2}(1 - (0-1)\sqrt{1+0})} = \frac{1}{2\sqrt{2}}$$ //

③ $f(x,y,z) = xyz, \quad \vec{p} = \langle x, y, z \rangle, \quad \vec{u} = \langle a, b, c \rangle$

Sol:

$$D_{\vec{u}} f(\vec{p}) = \lim_{h \to 0^+} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h} \cdot \frac{1}{|\vec{u}|}$$

$$= \lim_{h \to 0^+} \frac{f(x+ah, y+bh, z+ch) - f(x,y,z)}{h} \cdot \frac{1}{|\vec{u}|}$$

$$= \lim_{h \to 0^+} \frac{(x+ah)(y+bh)(z+ch) - xyz}{h} \cdot \frac{1}{|\vec{u}|}$$

$$= \lim_{h \to 0^+} \frac{\cancel{xyz} + \cancel{h}(xyc + xbz + xbch + ayz + aych + abzh + abch^2) - \cancel{xyz}}{\cancel{h}\ |\vec{u}|}$$

$$= \lim_{h \to 0^+} \left(xyc + xbz + ayz + h(xbc + ayc + abz + abch) \right) \cdot \frac{1}{|\vec{u}|}$$

$$= \frac{xyc + xbz + ayz + 0}{|\vec{u}|} = \frac{xyc + xbz + ayz}{\sqrt{a^2+b^2+c^2}}$$

④ $f(x,y) = 2x^2y, \quad \vec{p} = \langle x, y \rangle, \quad \vec{u}_1 = \langle 1, 0 \rangle$ and $\vec{u}_2 = \langle 0, 1 \rangle$

Sol:

$$D_{\vec{i}} f(x,y) = \lim_{h \to 0^+} \frac{f(x+h, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0^+} \frac{2(x+h)^2 y - 2x^2 y}{h}$$

$$= \lim_{h \to 0^+} \frac{(2x^2 + 4xh + 2h^2)y - 2x^2y}{h}$$

$$= \lim_{h \to 0^+} \frac{h(4x + 2h)y}{h}$$

$$= \lim_{h \to 0^+} (4x + 2h)y$$

$$= (4x + 0)y = 4xy$$

Note: y is constant here!

$$D_{\vec{j}} f(x,y) = \lim_{h \to 0^+} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \lim_{h \to 0^+} \frac{2x^2(y+h) - 2x^2y}{h}$$

$$= \lim_{h \to 0^+} \frac{2x^2h}{h}$$

$$= \lim_{h \to 0^+} 2x^2$$

$$= 2x^2$$

Note: x is constant here!

Partial Derivatives

The previous example illustrates an important case of directional derivatives. In particular, using a standard basis vector allows us to check the marginal change contributed by the corresponding variable.

Def^n: The k^{th} partial derivative of $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is

$$\frac{\partial f}{\partial x_k} := D_{\vec{e}_k} f(\vec{x})$$

partial derivative of f with respect to x_k.

$$= \lim_{h \to 0^+} \frac{f(\vec{x} + h\vec{e}_k) - f(\vec{x})}{h}$$

$$= \lim_{h \to 0^+} \frac{f(x_1, x_2, \cdots, x_k + h, \cdots, x_n) - f(x_1, x_2, \cdots, x_k, \cdots, x_n)}{h}$$

Note that this formula holds all variables except x_k constant. Thus we can think of $\frac{\partial}{\partial x_k}$ as taking usual derivatives of the single variable function $g(t) := f(x_1, x_2, \cdots, t, \cdots, x_n)$ and evaluating the result at x_k...

(t: k^{th} component)

$$g'(x_k) = \frac{\partial f}{\partial x_k}$$

Point:

① $\frac{\partial}{\partial x_k}$ treats $x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n$ as constants.

② $\frac{\partial}{\partial x_k}$ is otherwise a single-variable derivative $(\cdots)$.

③ $\frac{\partial}{\partial x_k}$ has all the usual properties of derivatives!

NB: We sometimes write $f_{x_k} = \frac{\partial f}{\partial x_k}$ for brevity of notation...

Ex: Compute all first-order partial derivatives of the function.

① $f(x,y) = x^2 + xy + y^3$

Sol 1 (Applying the definition):

$$\frac{\partial f}{\partial x} = D_{\vec{i}} f(x,y)$$
$$= \lim_{h\to 0^+} \frac{\left((x+h)^2 + (x+h)y + y^3\right) - (x^2+xy+y^3)}{h}$$
$$= \lim_{h\to 0^+} \frac{x^2 + 2xh + h^2 + xy + hy + y^3 - x^2 - xy - y^3}{h}$$
$$= \lim_{h\to 0^+} \frac{h(2x+h+y)}{h}$$
$$= \lim_{h\to 0^+} (2x+h+y)$$
$$= 2x + y$$

$$\frac{\partial f}{\partial y} = D_{\vec{j}} f(x,y)$$
$$= \lim_{h\to 0^+} \frac{\left(x^2 + x(y+h) + (y+h)^3\right) - (x^2+xy+y^3)}{h}$$
$$= \lim_{h\to 0^+} \frac{(x^2 + xy + xh + y^3 + 3hy^2 + 3h^2y + h^3) - (x^2 + xy + y^3)}{h}$$
$$= \lim_{h\to 0^+} \frac{h(x + 3y^2 + 3hy + h^2)}{h}$$
$$= \lim_{h\to 0^+} (x + 3y^2 + 3hy + h^2)$$
$$= x + 3y^2$$

//

Sol 2 (Take a normal derivative):

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2 + xy + y^3\right]$$
$$= \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[y^3]$$
$$= 2x + y + 0$$
$$= 2x + y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2 + xy + y^3\right]$$
$$= \frac{\partial}{\partial y}[x^2] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial y}[y^3]$$
$$= 0 + x + 3y^2$$
$$= x + 3y^2$$

//

② $f(x,y) = \ln(x^2+y^4)$

Sol: $\frac{\partial}{\partial x}\left[\ln(x^2+y^4)\right] = \frac{1}{x^2+y^4} \cdot \frac{\partial}{\partial x}[x^2+y^4] = \frac{2x}{x^2+y^4}$

$\frac{\partial}{\partial y}\left[\ln(x^2+y^4)\right] = \frac{1}{x^2+y^4} \cdot \frac{\partial}{\partial y}[x^2+y^4] = \frac{4y^3}{x^2+y^4}$

//

③ $f(x,y) = \sec(3x^2y - xy^2)$

Sol: $\frac{\partial}{\partial x}\left[\sec(3x^2y - xy^2)\right] = \sec(3x^2y - xy^2)\tan(3x^2y - xy^2)(6xy - y^2)$

$\frac{\partial}{\partial y}\left[\sec(3x^2y - xy^2)\right] = \sec(3x^2y - xy^2)\tan(3x^2y - xy^2)(3x^2 - 2xy)$

//

④ $f(x,y) = e^x \sin(y)$

Constant w.r.t x!

Sol: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^x \sin(y)\right] = \sin(y)\frac{\partial}{\partial x}\left[e^x\right] = e^x \sin(y)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^x \sin(y)\right] = e^x \frac{\partial}{\partial x}\left[\sin(y)\right] = e^x \cos(y)$

Constant w.r.t. y!

⑤ $f(x,y) = \sin((x+y)^2)$

Sol: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sin((x+y)^2)\right] = \cos((x+y)^2)\frac{\partial}{\partial x}\left[(x+y)^2\right] = \cos((x+y)^2)\cdot 2(x+y)$

Chain Rule

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\sin((x+y)^2)\right] = \cos((x+y)^2)\frac{\partial}{\partial y}\left[(x+y)^2\right] = \cos((x+y)^2)\cdot 2(x+y)$

⑥ $f(x,y,z) = xy^2 \arctan(z)$

Sol:

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy^2 \arctan(z)\right] = y^2 \arctan(z)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy^2 \arctan(z)\right] = 2xy \arctan(z)$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[xy^2 \arctan(z)\right] = \frac{xy^2}{1+z^2}$

⑦ $f(x,y,z) = e^{x^2+y^2}\sin(xz)\cos(yz)$

NB: Check!! :)

Sol:

$\frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = \cos(yz)\left(e^{x^2+y^2}\cos(xz)\cdot z + 2xe^{x^2+y^2}\sin(xz)\right)$ (Product rule)

$\frac{\partial}{\partial y}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = \sin(xz)\left(e^{x^2+y^2}\cdot -z\sin(yz) + 2ye^{x^2+y^2}\cos(yz)\right)$ (Product rule)

$\frac{\partial}{\partial z}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = e^{x^2+y^2}\left(-y\sin(xz)\sin(yz) + x\cos(xz)\cos(yz)\right)$ (Product rule)

Ex: Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for implicit function $x^2 + 2y^2 + 3z^2 = 1$.

Sol:

$$\frac{\partial}{\partial x}\left[x^2 + 2y^2 + 3z^2\right] = \frac{\partial}{\partial x}[1]$$

$$2x + 6z\frac{\partial z}{\partial x} = 0 \qquad \therefore \frac{\partial z}{\partial x} = -\frac{x}{3z}$$

$$\frac{\partial}{\partial y}\left[x^2 + 2y^2 + 3z^2\right] = \frac{\partial}{\partial y}[1]$$

$$4y + 6z\frac{\partial z}{\partial y} = 0 \qquad \therefore \frac{\partial z}{\partial y} = -\frac{2y}{3z}$$

$\blacksquare$

Ex: Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for $f(x,y,z) = \frac{x}{y+z^2}$ using the definition of the partial derivative.

Sol:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{\frac{x+h}{y+z^2} - \frac{x}{y+z^2}}{h} = \lim_{h\to 0} \frac{1}{h}\cdot\frac{h}{y+z^2} = \lim_{h\to 0}\frac{1}{y+z^2} = \frac{1}{y+z^2}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{\frac{x}{(y+h)+z^2} - \frac{x}{y+z^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{x(y+z^2) - x(h+(y+z^2))}{((y+h)+z^2)\cdot(y+z^2)}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{-xh}{((y+h)+z^2)(y+z^2)}$$

$$= \lim_{h\to 0} -\frac{x}{((y+h)+z^2)(y+z^2)}$$

$$= -\frac{x}{(y+z^2)^2}$$

$$\frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{\frac{x}{y+(z+h)^2} - \frac{x}{y+z^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{x\left((y+z^2) - (y+(z+h)^2)\right)}{(y+(z+h)^2)(y+z^2)}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{x(z^2 - z^2 - 2zh - h^2)}{(y+(z+h)^2)(y+z^2)}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{hx(-2z-h)}{(y+(z+h)^2)(y+z^2)}$$

$$= \lim_{h\to 0} \frac{-x(2z+h)}{(y+(z+h)^2)(y+z^2)}$$

$$= -\frac{2xz}{(y+z^2)^2}$$

$\blacksquare$

Ex: Compute all second order derivatives of the function.

① $f(x,y,z) = x^2 + 7yz - 2x^2yz^3$

Sol: $\dfrac{\partial f}{\partial x} = 2x - 4xyz^3$ $\quad \dfrac{\partial f}{\partial y} = 7z - 2x^2z^3$ $\quad \dfrac{\partial f}{\partial z} = 7y - 6x^2yz^2$

second applied \ first applied	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
$\frac{\partial}{\partial x}$	$\frac{\partial^2 f}{(\partial x)^2} = 2 - 4yz^3$	$\frac{\partial^2 f}{\partial x \partial y} = -4xz^3$	$\left(\frac{\partial^2 f}{\partial x \partial z} = -12xyz^2\right)$
$\frac{\partial}{\partial y}$	$\frac{\partial^2 f}{\partial y \partial x} = -4xz^3$	$\frac{\partial^2 f}{(\partial y)^2} = 0$	✗ $\frac{\partial^2 f}{\partial y \partial z} = 7 - 6x^2z^2$
$\frac{\partial}{\partial z}$	$\left(\frac{\partial^2 f}{\partial z \partial x} = -12xyz^2\right)$	✗ $\frac{\partial^2 f}{\partial z \partial y} = 7 - 6x^2z^2$	$\frac{\partial^2 f}{(\partial z)^2} = -12x^2yz$

② $g(x,y) = \sin(xy)\cos(x^2+y^2)$

Sol: Left to you as an exercise (verify $\dfrac{\partial^2 g}{\partial y \partial x} = \dfrac{\partial^2 g}{\partial x \partial y}$).

Prop (Clairaut's Theorem): Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function with $(a,b) \in D$ for D a disk. If the second-order mixed partial derivatives f_{xy} and f_{yx} are defined and continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

pf: Define two new functions of a single variable:

$$\Delta(h) := \big(f(a+h,b+h) - f(a+h,b)\big) - \big(f(a,b+h) - f(a,b)\big) \quad (\text{for } (a+h,b+h) \in D)$$

$$\alpha(x) := f(x,b+h) - f(x,b) \quad (\text{for } h \neq 0)$$

Now for all $h \neq 0$ satisfying $(a+h,b+h) \in D$ we obtain a constant c_h satisfying $|a - c_h| \leq |h|$ such that

$$h\,\alpha'(c_h) = \alpha(a+h) - \alpha(a)$$

$f'(c) = \frac{f(b)-f(a)}{b-a}$ (see Calculus 1 notes).

by the Mean Value Theorem. Thus we compute

$$\begin{aligned}\Delta(h) &= \big(f(a+h,b+h) - f(a+h,b)\big) - \big(f(a,b+h) - f(a,b)\big)\\ &= \alpha(a+h) - \alpha(a)\\ &= h\,\alpha'(c_h)\\ &= h\big(f_x(c_h,b+h) - f_x(c_h,b)\big) \quad \star\end{aligned}$$

Now define $\beta_h(y) := f_x(c_h, y)$.

Again apply the Mean Value Theorem to obtain a constant d_h satisfying $|b - d_h| \leq |h|$ such that

$$h\beta_h'(d_h) = \beta_h(b+h) - \beta_h(b) = f_x(c_h, b+h) - f_x(c_h, b)$$

Thus we can use our above work to compute

$$\begin{aligned}\Delta(h) &= h\left(f_x(c_h, b+h) - f_x(c_h, b)\right)\\ &= h^2 \beta'(d_h)\\ &= h^2 f_{xy}(c_h, d_h)\end{aligned}$$

Now we could also rewrite

$$\Delta(h) = \left(f(a+h, b+h) - f(a, b+h)\right) - \left(f(a+h, b) - f(a,b)\right)$$

Thus a very similar argument to the above yields

$$\Delta(h) = h^2 f_{yx}(\gamma_h, \delta_h)$$

Where $|\gamma_h - a|, |\delta_h - b| \leq |h|$ for all $h \neq 0$.

Hence we have the following equality for all $h \neq 0$ with $|h|$ sufficiently small:

$$\star\ f_{xy}(c_h, d_h) = \frac{\Delta(h)}{h^2} = f_{yx}(\gamma_h, \delta_h)$$

By our construction $\lim_{h\to 0}(c_h, d_h) = (a,b) = \lim_{h \to 0}(\gamma_h, \delta_h)$.

Hence we obtain the following by continuity of f_{xy} and f_{yx}:

$$\begin{aligned} f_{xy}(a,b) &= f_{xy}\left(\lim_{h\to 0}(c_h, d_h)\right) = \lim_{h \to 0} \underline{f_{xy}(c_h, d_h)}\\ &\overset{\star}{=} \lim_{h\to 0} f_{yx}(\gamma_h, \delta_h) = f_{yx}\left(\lim_{h\to 0}(\gamma_h, \delta_h)\right) = f_{yx}(a,b). \quad \square\end{aligned}$$

Cor: If $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ has continuous mixed partial derivatives of order d, then these are equal provided they have the same symbols.

Point: For sufficiently nice functions, mixed partial derivatives only depend on the variables with respect to which we differentiate, and not the order in which we applied the partial derivatives. Thus (e.g.)

$$\frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = \frac{\partial^3 f}{\partial x\, \partial z\, \partial y} = \frac{\partial^3 f}{\partial y\, \partial x\, \partial z} = \frac{\partial^3 f}{\partial y\, \partial z\, \partial x} = \frac{\partial^3 f}{\partial z\, \partial x\, \partial y} = \frac{\partial^3 f}{\partial z\, \partial y\, \partial x}$$

for a function of n variables with all of the above continuous.

Ex: Compute all 3rd order partial derivatives of

$$f(x,y,z) = e^x \sin(yz).$$

Sol:

1: $\frac{\partial f}{\partial x} = e^x \sin(yz)$ $\quad \frac{\partial f}{\partial y} = z e^x \cos(yz)$ $\quad \frac{\partial f}{\partial z} = y e^x \cos(yz)$

2: $\frac{\partial^2 f}{(\partial x)^2} = e^x \sin(yz)$ $\quad \frac{\partial^2 f}{\partial x \partial y} = z e^x \cos(yz)$ $\quad \frac{\partial^2 f}{\partial x \partial z} = y e^x \cos(yz)$

$\frac{\partial^2 f}{(\partial y)^2} = -z^2 e^x \sin(yz)$ $\quad \frac{\partial^2 f}{\partial y \partial z} = e^x\left(-yz \sin(yz) + \cos(yz)\right)$ $\quad \frac{\partial^2 f}{(\partial z)^2} = -y^2 e^x \sin(yz)$

3: $\frac{\partial^3 f}{(\partial x)^3} = e^x \sin(yz)$ $\quad \frac{\partial^3 f}{(\partial x)^2 \partial y} = z e^x \cos(yz)$ $\quad \frac{\partial^3 f}{(\partial x)^2 \partial z} = y e^x \cos(yz)$

$\frac{\partial^3 f}{\partial x (\partial y)^2} = -z e^x \sin(yz)$ $\quad \frac{\partial^3 f}{(\partial y)^3} = -z^3 e^x \cos(yz)$ $\quad \frac{\partial^3 f}{(\partial y)^2 \partial z} = -z e^x \left(zy \cos(yz) + 2\sin(yz)\right)$

$\frac{\partial^3 f}{\partial x (\partial z)^2} = -y^2 e^x \sin(yz)$ $\quad \frac{\partial^3 f}{\partial y (\partial z)^2} = -y e^x\left(yz \cos(yz) + 2\sin(yz)\right)$ $\quad \frac{\partial^3 f}{(\partial z)^3} = -y^3 e^x \cos(yz)$

$$\frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = e^x\left(-yz \sin(yz) + \cos(yz)\right)$$

How many?

BC $\begin{cases} 1: 3 \\ 2: 6 + 3 \\ 3: 3 + 18 + 6 \end{cases}$

AC $\begin{cases} 1: 3 \\ 2: 6 \\ 3: 10 \end{cases}$

$\blacksquare$